Kinematic Shakedown Analysis of Structures with Cohesive-Frictional Materials

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Abstract: The purpose of this paper is to develop a general nonlinear approach for shakedown analysis of structures with cohesive-frictional materials. Then the limit status and load of this kind of structures subject to repeated loads can be solved. By introducing a general cohesive-frictional constitutive model into the kinematic shakedown theorem, the shakedown criterion for this kind of materials is established. With aid of the mathematical programming theory and the finite element method, a nonlinear programming formulation with only a small number of equality constraints is obtained. In order to more efficiently solve this problem for some specific materials, several explicit formulations are given. And a direct iterative algorithm is proposed to solve the resulting formulation. Numerical applications to the shakedown analysis of offshore pipelines demonstrate that the developed method can efficiently calculate the effect of cyclic loading, e.g. resulting from environmental loads or operation loads, on the stability load of offshore structures.

Keywords: Shakedown analysis, nonlinear programming, finite element method, kinematic shakedown theorem, cohesive frictional materials.

1. INTRODUCTION

Shakedown analysis is often used to solve the plastic limit status of a structure under cyclic or repeated loading. Then, the plastic limit load (shakedown limit) can then be obtained. There are three phenomena which may occur for a structure under cyclic or repeated loading. The first phenomenon is the purely elastic deformation, when the applied load magnitude is lower than the elastic limit of structures. The second phenomenon is the shakedown status when the load is larger than the elastic limit of structures but less than a critical limit – the shakedown limit. The third phenomenon is the non-shakedown status when the load is higher than the shakedown limit, a non-restricted plastic flow will occur. If the non-shakedown phenomenon happens, the structure may undergo the failure mode of either incremental collapse (ratchetting) or alternating plasticity (low cycle fatigue). In the failure of incremental collapse, plastic strain increases with the successive loading cycles and an unlimited plastic flow finally happens until the structure fails. In the failure of alternating plasticity, the material at the plastic region intends to be damaged due to low-cycle fatigue and high plastic work density, but the plastic deformation still remains small. By means of shakedown analysis, the non-shakedown condition can be found. And the corresponding limit load (shakedown limit) can be determined. Mostly, the classical shakedown analysis is implemented by means of a direct plasticity approach.

Shakedown analysis is based on two fundamental shakedown theorems, i.e. the static, lower bound theorem [1] and the kinematic, upper bound theorem [2]. The most of the early studies on shakedown analysis focused on the analytical solutions for some simple structures. Later, numerical techniques drew more attentions for more complex engineering structures. In the last two decades, with the rapid development of computational techniques, the numerical methods for shakedown analysis have been developed rapidly [3-22]. However, most of these works focused on frictionless materials.

The cohesive-frictional materials are widely used in modern industry, e.g. the polymeric material is a typical cohesive-frictional material used in automotive, aerospace and offshore engineering. For these kinds of materials, they normally exhibit complicated material behaviour, e.g. non-standard material constitutive models for their yielding and plastic flow. Due to the highly nonlinear and non-smooth mathematical expression, it is still a challenging work to numerically implement this kind of materials. So even in the traditional incremental elastoplastic analysis, this difficulty is not fully solved. It will become much more difficult to implement shakedown analysis for this kind of materials. Up to now, there is only a few literatures which discussed this topic. Li [12] developed a nonlinear programming approach to implement shakedown analysis for the cohesive-frictional microstructures.

This paper is to develop a general numerical method for shakedown analysis of a macrostructure with the cohesive-frictional material. Moreover, several
explicit expressions for the associated plastic flow materials are given. The proposed method is based on the kinematic shakedown theorem and then an upper bound to the plastic limit load of a structure is obtained. With aid of the mathematical programming theory and the finite element method, the numerical model is finally formulated as a nonlinear programming problem subject to only a small number of equality constraints. A direct iterative algorithm is then proposed to solve it. Numerical examples further indicate the efficiency of the proposed method.

2. KINEMATIC SHAKEDOWN ANALYSIS

2.1. Kinematic Shakedown Theorem

Shakedown analysis is based on an elastic-perfectly plastic material model. In this paper, a general expression for the cohesive-frictional material is used. In order to implement the finite element method, the column vectors are used for the expression of true stress \( \sigma \), true strain \( \varepsilon \), and displacement \( u \), e.g. 
\[
\sigma = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sqrt{2}\sigma_{12}, \sqrt{2}\sigma_{23}, \sqrt{2}\sigma_{13}]^T, \quad \varepsilon = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \sqrt{2}\varepsilon_{12}, \sqrt{2}\varepsilon_{23}, \sqrt{2}\varepsilon_{13}]^T, \quad u = [u_1, u_2, u_3]^T. 
\]
A structure of a body \( V \) with boundary \( \Gamma \), experiences cyclic loads on the force boundary \( \Gamma_1 \) and is fixed for the other boundary denoted by \( \Gamma_0 \).

When the kinematic shakedown technique is used, an upper bound to the shakedown limit is obtained. The kinematic shakedown theorem [2] states: shakedown occurs for a structure under repeated or cyclic loads, when the rate of plastic dissipation power exceeds the work rate of external forces for any admissible plastic strain-rate cycles and all loading paths. In other words, non-shakedown occurs, if the rate of plastic dissipation power is less than the work rate of external forces for any one admissible plastic strain-rate cycle or any one loading path.

Then, the kinematic shakedown theorem can be formulated as;

\[
\lambda_{\omega} \int_0^T \left( \int_{\Gamma} t_i \dot{u}_i^* \, d\Gamma + \int_V f_i \dot{u}_i^* \, dv \right) \, dt \leq \int_0^T \int_V D(\dot{\varepsilon}_p^*^n) \, dv \, dt \quad (1) 
\]

where \( \lambda_{\omega} \) is the shakedown limit multiplier, \( t_i \) is the applied reference load of tractions, \( f_i \) is the applied reference load of body forces, \( t_i \) and \( f_i \) are cyclic over a time interval \( [0, T] \), \( \dot{u}_i \) is the displacement velocity, \( \dot{\varepsilon}_p^* \) is the plastic strain rate, "\( D(\dot{\varepsilon}_p^*^n) \)" is a function for the rate of plastic dissipation power in terms of the admissible strain rate, the superscript \( ^n \) stands for a parameter corresponding to the kinematically admissible strain field, \( \Gamma_1 \) denotes the force boundary, and \( V \) represents the space domain of the structure.

By means of the virtual work, the first term in the left-hand side of Equation (1) can be rewritten as

\[
\int_0^T \int_{\Gamma} t_i \dot{u}_i^* \, d\Gamma \, dt = \int_0^T \int_V \sigma_{ij}^* \dot{\varepsilon}_{ij}^* \, dv \, dt + \int_0^T \int_V \sigma_{ij}^* \dot{\varepsilon}_{ij}^* \, dv \, dt 
\]

where \( \sigma_{ij}^* \) is the elastic stress from the force \( t_i \), \( \sigma_{ij}^* \) is theelastic compliance tensor of the material, \( \dot{\varepsilon}_{ij}^* \) is the residual stress rate associated with the admissible plastic strain rate \( \dot{\varepsilon}_{ij}^* \), and \( \dot{\varepsilon}_{ij}^* \) is self-equilibrated.

If the body force is omitted, the equation (1) becomes as;

\[
\lambda_{\omega} \int_0^T \int_V \sigma_{ij}^* \dot{\varepsilon}_{ij}^* \, dv \, dt \leq \int_0^T \int_V \sigma_{ij}^* \dot{\varepsilon}_{ij}^* \, dv \, dt \quad \text{s.t.} \quad \Delta \varepsilon_{ij}^* = \frac{1}{2} (\Delta u_i + \Delta u_j) \quad \text{in} \quad V \\
\Delta u_i = \int_0^T \dot{u}_i \, dt \quad \text{in} \quad V \\
\Delta u_i = 0 \quad \text{on} \quad \Gamma_0 
\]

where \( \Delta \varepsilon_{ij}^* \) and \( \Delta u_i \) are the cumulative plastic strain and displacement at the end of one loading cycle.

2.2. A Cohesive-Frictio nal Material Model

For the purpose of the general implementation for the shakedown analysis, the following expression for the yield criterion of the cohesive-frictional material is adopted.

\[
F(I_1, J_2) = \phi_0 I_1 + \sqrt{J_2} c_0 = 0 \quad (4) 
\]

where \( I_1 \) is the first invariant of stress tensors, \( J_2 \) is the second invariant of deviatoric stress tensors, \( \phi_0 \) and \( c_0 \) are strength parameters of the material.

When the applied force is beyond the yield stress of the material, plastic flow and plastic deformation will occur. Then, a plastic flow rule is required to describe the evolution of deformation. The plastic flow rule defines both the direction and magnitude of the plastic strain-rate, and is normally expressed by a plastic potential \( \omega(\sigma) \).

\[
\dot{\varepsilon}^p = \mu^p \frac{d\omega(\sigma)}{d\sigma} \quad (5) 
\]
where \( \omega(\sigma) \) denotes a plastic potential function that resembles the yield function \( F(\sigma) \), and \( \mu \) is a non-negative plastic proportionality factor. If the associated plastic flow is assumed for a material, \( \omega(\sigma) \) will be equivalent to \( F(\sigma) \).

For most of the current materials, a second-order polynomial can capture the plastic flow as;

\[
\Omega(\sigma) = \sigma^T P^P \sigma + \sigma^T Q^P - 1 = 0
\]  

(6)

where \( P^P \) and \( Q^P \) are the coefficient matrices of the plastic flow, and \( P^P \) is symmetric.

Based on the yield criterion (4) and the plastic potential (6), the plastic proportionality factor and the plastic dissipation power can be determined as follows.

\[
\dot{\mu} = \frac{\dot{\varepsilon}^T (P^P)^{-1} P^P \dot{\varepsilon}^T}{\sqrt{\dot{\varepsilon}^T (P^P)^{-1} (4 P^P + 4 Q^P)^{-1} (P^P)^{-1} \dot{\varepsilon}^T}}
\]  

(7)

\[
D(\dot{\varepsilon}^T) = \sigma^T \dot{\varepsilon}^T = \sigma^2 \dot{\varepsilon}^T
\]

\[
= \left( \frac{1}{2 \mu} (P^P)^{-1} \dot{\varepsilon}^T - \frac{1}{2} (P^P)^{-1} Q^P \right)^T \dot{\varepsilon}^T
\]

\[
= \frac{1}{2 \mu} (\dot{\varepsilon}^T)^2 (P^P)^{-1} \dot{\varepsilon}^T - \frac{1}{2} (Q^P)^T (P^P)^{-1} \dot{\varepsilon}^T
\]  

(8)

where \( P \) and \( Q \) are coefficient matrices and related to the strength properties of the material which are defined in Equation (4).

2.3. Nonlinear Mathematical Programming

In order to calculate the time integration in the kinematic shakedown analysis (3), the numerical technique proposed in the literatures [23, 24] is used in this paper. The accumulation of the plastic strain and the plastic flow power is computed as follows.

\[
\Delta \dot{\varepsilon} = \sum_{t_{i-1}}^{t_i} \dot{\varepsilon}^P dt
\]  

(9)

\[
\mu = \int_{t_{i-1}}^{t_i} \dot{\mu} dt.
\]  

(10)

Then, by means of the mathematical programming theory and the expression of the plastic dissipation power (8), the kinematic shakedown analysis (3) can formulated as the following the minimum optimization problem with equality constraints.

\[
\tilde{\lambda}_{sd} = \min_{\Delta \varepsilon^p} \sum_{t_{i-1}}^{t_i} \int_{V} \left( \frac{\dot{\varepsilon}^T}{2} \left( \sqrt{\dot{\varepsilon}^T R^P e_T^p + T^P} \right) + \frac{1}{2} (Q^P)^T (P^P)^{-1} \dot{\varepsilon}^T \right) dv
\]

s. t. \[ \sum_{t_{i-1}}^{t_i} \int_{V} (\sigma_i^T)^T e_p^i dv = 1 \]

\[
\Delta \varepsilon^p = \sum_{i=1}^{n} e_p^i = \Psi(\Delta u) \quad \text{in} \ V
\]

\[
\Delta u = 0 \quad \text{on} \ G_u
\]

(11)

where the coefficient matrices are defined as;

\[
\delta^p = (e_p^i)^T (P^P)^{-1} e_p^i
\]  

(12)

\[
R^T = (P^P)^{-1} P (P^P)^{-1}
\]  

(13)

\[
R^t = (Q^P)^T (P^P)^{-1}
\]  

(14)

\[
T^T = (Q^P)^T - (Q^P)^T (P^P)^{-1}
\]  

(15)

Finally, the kinematic shakedown analysis is formulated as a nonlinear mathematical programming problem subject to equality constraints. The solution of the above minimum optimization problem is to find the shakedown multiplier \( \tilde{\lambda}_{sd} \), with \( \tilde{\lambda}_{sd} \cdot t_i \) being the shakedown limit of structures.

2.4. Explicit Solutions

The solution (11) for the kinematic shakedown analysis is an implicit form. In order to explicitly give the expression for a specific material. More material model parameters are normally required so that the expression (11) can be simplified.

For an anisotropic material, e.g. the Tasi-Wu and Hill yielding materials with the associated plastic flow, the explicit expression of the kinematic shakedown analysis can be expressed as;

\[
\tilde{\lambda}_{sd} = \min_{\Delta \varepsilon^p} \sum_{t_{i-1}}^{t_i} \int_{V} \left( \frac{\delta^p}{2} \left( \sqrt{\dot{\varepsilon}^T R^P e_T^p + T^P} \right) + \frac{1}{2} (Q^P)^T (P^P)^{-1} \dot{\varepsilon}^T \right) dv
\]

s. t. \[ \sum_{t_{i-1}}^{t_i} \int_{V} (\sigma_i^T)^T e_p^i dv = 1 \]

\[
\Delta \varepsilon^p = \sum_{i=1}^{n} e_p^i = \Psi(\Delta u) \quad \text{in} \ V
\]

\[
\Delta u = 0 \quad \text{on} \ G_u
\]

(16)

For a pressure-dependent material, e.g. the Mohr-Coulomb and Drucker-Prager yielding materials with the associated plastic flow, the explicit expression of the kinematic shakedown analysis can be expressed as;

\[
\tilde{\lambda}_{sd} = \min_{\Delta \varepsilon^p} \sum_{t_{i-1}}^{t_i} \int_{V} \left( \frac{\delta^p}{2} \left( \sqrt{\dot{\varepsilon}^T R^P e_T^p + T^P} \right) + \frac{1}{2} (Q^P)^T (P^P)^{-1} \dot{\varepsilon}^T \right) dv
\]

s. t. \[ \sum_{t_{i-1}}^{t_i} \int_{V} (\sigma_i^T)^T e_p^i dv = 1 \]

\[
\Delta \varepsilon^p = \sum_{i=1}^{n} e_p^i = \Psi(\Delta u) \quad \text{in} \ V
\]

\[
\Delta u = 0 \quad \text{on} \ G_u
\]

(17)
3. NUMERICAL MODELLING

3.1. Finite Element Discretization

In order to numerically implement the kinematic shakedown analysis, the finite element method is used. Based on the finite element technique, a structure is discretized by finite elements and then the displacement velocity and strain rate can be interpolated in terms of an unknown nodal displacement velocity vector.

\[ \Delta u(x) = N(x) \Delta \delta, \]

\[ \Delta e(x) = B(x) \Delta \delta, \]

where, with reference to the \( e \)-th finite element, \( \Delta \delta_e \) is the nodal cumulative displacement column vector over a loading cycle, \( N_e \) is the interpolation function, and \( B_e \) is the strain function.

Then, the finite element form of the kinematic shakedown analysis (11) can be expressed as;

\[
\begin{align*}
\lambda_e &= \min \sum_{k=1}^{n} \rho_k |J_k| \left( \frac{\delta^f}{2} \left( \sqrt{|e_{e_k}^p|^2 + T \delta_{e_k}^p} \right)^2 \right) - \frac{1}{2} (Q)^T (P)^t e_{e_k}^p \\
\text{s.t.} \quad \sum_{k=1}^{n} \rho_k |J_k| (\sigma_{e_k}^p)^t e_{e_k}^p &= 1 \\
\Delta e_{e_k}^p &= \sum_{k=1}^{n} e_{e_k}^p = B_e \Delta \delta & (r = 1, 2, \cdots, n)
\end{align*}
\]

(20a,b,c)

where, with reference to the \( r \)-th Gaussian integral point, \( \rho_r \) is the Gaussian integral weight, \( |J_k| \) is the determinant of the Jacobian matrix, \( n \) is the number of the Gaussian integral points within the FE-discretized structure, and \( B_e \) is the strain matrix at the \( r \)-th Gaussian integral point.

3.2. A Direct Iterative Algorithm

The kinematic shakedown analysis (20) is a nonlinear mathematical programming problem subject to equality constraints. The objective function is nonlinear, continuous but non-differentiable in the region where the item of the square root calculation equals to zero. This causes some difficulties in solving the programming problem. For a linear non-differentiable programming problem, if the objective function is finite and continuous in a feasible set, it is not necessary to be differentiable everywhere and an optimal solution can be obtained [25]. An iterative algorithm was proposed to overcome the similar numerical difficulty [12, 13], where a technique based on distinguishing elastic/plastic areas was developed. It is extended in this paper to solve the nonlinear mathematical programming problem (20).

First, based on the mathematical programming theory [26, 27], the equality constraints of the normalization condition and the geometric compatibility are introduced into the optimization problem by means of the Lagrangian method. As a result, an unconstrained minimum optimization problem is obtained as follows.

\[ \Pi(e_0, \Delta \delta, L', I') = \sum_{k=1}^{n} \rho_k |J_k| \left( \frac{\delta^f}{2} \left( \sqrt{|e_{e_k}^p|^2 + T \delta_{e_k}^p} \right)^2 \right) - \frac{1}{2} (Q)^T (P)^t e_{e_k}^p \]

(21)

\[ + L' \left( \sum_{k=1}^{n} \rho_k |J_k| (\sigma_{e_k}^p)^t e_{e_k}^p \right) + \sum_{r=1}^{n} L_r \left( \sum_{k=1}^{n} e_{e_k}^p - B_e \Delta \delta \right) \]

where \( L' \) and \( I' \) are Lagrangean multipliers.

Then, according to the Kuhn-Tucker stationarity condition, the following set of equations can be obtained to solve the kinematic shakedown analysis problem (21).

\[
\begin{align*}
\delta^f (H_{e_k})_{e_k} e_{e_k}^p - \frac{\delta^f}{2} (T^t)^{-1} - L_k \sigma_{e_k}^p + (\rho_k |J_k|)^{-1} L_r - \frac{1}{2} (P^t)^t Q^t &= 0, \\
(k = 1, 2, \cdots, l; r = 1, 2, \cdots, n) \\
\sum_{r=1}^{n} (B^t L_r) &= 0 \\
\sum_{k=1}^{n} \rho_k |J_k| (\sigma_{e_k}^p)^t e_{e_k}^p &= 1 \\
\sum_{r=1}^{n} e_{e_k}^p - B_e \Delta \delta &= 0 & (r = 1, 2, \cdots, n)
\end{align*}
\]

(22)

where \( H_{e_k} \) is the coefficient matrix and defined as;

\[
(H_{e_k})_{e_k,p} = \frac{1}{2} R^t (z_{e_k})_{e_k,p}^t \quad z_{e_k} = \sqrt{|e_{e_k}^p|^2 + T \delta_{e_k}^p}.
\]

(23a,b)

By solving the set of equations (22), the variable \( e_{e_k}^p \) can be determined. Then the shakedown load multiplier can be calculated as;

\[
\lambda_e = \sum_{k=1}^{n} \sum_{l=1}^{n} \rho_k |J_k| \left( \frac{\delta^f}{2} \left( \sqrt{|e_{e_k}^p|^2 + T \delta_{e_k}^p} \right)^2 \right) - \frac{1}{2} (Q)^T (P)^t e_{e_k}^p
\]

(24)

In order to trigger the iteration, the iteration seeds are defined as follows.

\[
(z_{e_k})_{e_k} = 1 \quad (k = 1, 2, \cdots, l; r = 1, 2, \cdots, n), \quad (\delta^f)_{e_k} = 1.0
\]

(25a,b)

The iteration starts with the hypothesis that the whole structure is in the plastic state, which has \( (H_{e_k})_{e_k,p} = \frac{1}{2} R^t. \)
Based on the computational results at the iterative step \( h \), the value of \( z_{kr} \) \((z_{kr} = \sqrt{v^2 + w^2})\) needs to be calculated at every Gaussian integral point to check whether it is in a non-differentiable area. Then the Gaussian integral point set \( I \) will be subdivided into two subsets: a subset \((I_E)_{h+1}\) where the objective function is not differentiable, and a subset \((I_P)_{h+1}\) where the objective function is differentiable.

Once a non-differentiable region is found, the objective function in Eq. (44) will be modified by removing the calculation for this non-differentiable region. Correspondingly, a constraint, that the value of \( z_{kr} \) is equal to zero in this region, will be introduced into the mathematical programming problem by the penalty function method. Then, the coefficient matrix \( H_{kr} \) at this iteration step will be updated as

\[
(H_{kr})_{h+1} = \begin{cases} \beta R^k & r \in (I_E)_{h+1} \\ \frac{1}{2} R^k(z_{kr})^{-1} & r \in (I_P)_{h+1} \end{cases}
\]  

(26)

where \( \beta \) is the penalty factor which is used to introduce the constraint of non-differentiable areas into the programming problem. In practice, the typical value of \( \beta \) is from \( 10^6 \) to \( 10^{12} \).

Once the iteration starts, it will be repeated until the following convergence criteria are satisfied.

\[
\begin{align*}
|\lambda_{k+1} - \lambda_k| & \leq \eta_1 \\
||\Delta \lambda_{k+1} - \Delta \lambda_k|| & \leq \eta_2
\end{align*}
\]

(27)

where \( \eta_1 \) and \( \eta_2 \) are computational error tolerances.

Once the convergence condition (27) is attained, the shakedown limit multiplier at the current iterative step is the optimized solution.

4. APPLICATIONS

Pipelines are widely used in many industries. For example, in offshore engineering, sub sea pipes, tendons for offshore floating platforms, and TTR/SCR risers are the typical pipeline structures. For these kinds of structures, they are often subject to cyclic or repeated loads. The stability condition and the bearing

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**Figure 1:** A multi-layered pipeline under combined loads.

**Figure 2:** The cross section of the three layered pipeline.
Kinematic Shakedown Analysis of Structures with Cohesive-Frictional Capacity of structures are the key design parameters. The developed method can provide a robust analysis tool for the strength computation. In this section, by means of the developed numerical approach, the stability and shakedown load of pipelines under cyclic loading is calculated.

Here, a multi-layered pipeline under combined loads is analyzed to find its stability condition and bearing capacity. The cross sections of the multi-layered pipeline are given in Figures 1 and 2. The geometric size of the multi-layered pipeline for the numerical simulation is chosen as: \( L_0 = 5.0\, \text{m} \), \( R_1 = 0.35\, \text{m} \), \( R_2 = 0.4\, \text{m} \), \( R_3 = 0.45\, \text{m} \), \( R_4 = 0.5\, \text{m} \). The material parameters of the three layers are given in Table 1.

### Table 1: The Material Parameters of the Three Layered Pipeline

<table>
<thead>
<tr>
<th>Layer No.</th>
<th>Material Model</th>
<th>Yielding Parameters</th>
<th>Yielding Criterion</th>
<th>Elastic Modulus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal layer</td>
<td>( \sigma_s (\text{Pa}) )</td>
<td>von Mises</td>
<td>200 (GPa)</td>
<td></td>
</tr>
<tr>
<td>Middle layer</td>
<td>( c_0 = 0.5\sigma_s ), ( \phi_0 = 15^\circ )</td>
<td>Drucker-Prager</td>
<td>100 (GPa)</td>
<td></td>
</tr>
<tr>
<td>External layer</td>
<td>( \sigma_s (\text{Pa}) )</td>
<td>von Mises</td>
<td>200 (GPa)</td>
<td></td>
</tr>
</tbody>
</table>

The pipeline is subject to internal pressure, axial tension and bending moment which are denoted by \( P \), \( N \) and \( M \). An additional axial tension \( N_1 \) induced by the internal pressure \( P \) is considered, i.e., \( N_1 = P \pi R_1^2 \), and the bending moment \( M \) can be simulated by a couple of tensile and compression at the end of pipeline. Due to the symmetry of the structure and the external loads, one quarter of the structure is numerically calculated and is discretized by three-dimensional eight-node isoparametric finite elements. The convergence tolerances are chosen as \( \eta_1 = \eta_2 = 10^{-3} \) in the numerical iterative algorithm.

In order to non-dimensionally present the numerical results, the following non-dimensional loads are defined.

\[
p = \frac{P}{P_0}, \quad n = \frac{N}{N_0}, \quad m = \frac{M}{M_0}
\]

(28a, b, c)

where \( P_0 \), \( N_0 \) and \( M_0 \) denote the limit load of the structure under a single load of the internal pressure, the axial tension or the bending moment. By means of the developed numerical method, the single shakedown limits are obtained as follows.

\[
\begin{align*}
\sigma_s &= 0.411 \sigma_s \text{ (MPa)} \\
N_0 &= 0.362 \sigma_s \text{ (N)} \\
M_0 &= 0.102 \sigma_s \text{ (N\cdotm)}
\end{align*}
\]

The shakedown limit of the multi-layered pipeline under combined cyclic loads is calculated and the

![Figure 3: Shakedown limit of the pipeline under internal pressure and axial tension.](image3)

![Figure 4: Shakedown limit of the pipeline under internal pressure and bending moment.](image4)

![Figure 5: Shakedown limit of the pipeline under axial tension and bending moment.](image5)
numerical results are presented in Figures 3-5. The shakedown limit is an important parameter for the stability analyses of engineering structures. Based on these results, a more reliable design can be guaranteed for a structure under cyclic loading.

CONCLUSIONS

A novel nonlinear mathematical programming method is developed in this paper to implement the kinematic shakedown analysis for the cohesive-frictional material. The developed method is based on a purely kinematical form, so that the computational effort is very modest. Moreover, the nonlinear yield surface is not linearized and introduced into shakedown analysis. Therefore, the numerical precision is very high. The proposed numerical iterative algorithm can efficiently overcome the difficulty that the objective function is non-differentiable. The numerical examples also show the numerical efficiency and stability of the proposed algorithm. The proposed approach provides a general methodology for the stability analysis and the bearing capacity calculation of structures under cyclic or repeated loading.

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